

# A NOTE ON THE DERIVATION OF RIGID-PLASTIC MODELS

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**ABSTRACT.** This note is devoted to a rigorous derivation of rigid-plasticity as the limit of elasto-plasticity when the elasticity tends to infinity.

## 1. INTRODUCTION

Small strain elasto-plasticity is formally modeled as follows. Consider a homogeneous elasto-plastic material occupying a volume  $\Omega \subset \mathbb{R}^n$  with Hooke's law (elasticity tensor)  $\mathbb{C}$ . Assume that the body is subjected to a time-dependent loading process during a time interval  $[0, T]$  with, say,  $f(t)$  as body loads,  $g(t)$  as surface loads on a part  $\Gamma_N$  of  $\partial\Omega$ , and  $w(t)$  as displacement loads (hard device) on the complementary part  $\Gamma_D$  of  $\partial\Omega$ . Denoting by  $Eu(t)$  the infinitesimal strain at  $t$ , that is, the symmetric part of the spatial gradient of the displacement field  $u(t)$  at  $t$ , small strain elasto-plasticity requires that  $Eu(t)$  decompose additively as

$$Eu(t) = e(t) + p(t) \text{ in } \Omega, \text{ with } u(t) = w(t) \text{ on } \Gamma_D$$

where  $e(t)$  is the elastic strain and  $p(t)$  the plastic strain. The elastic strain is related to the stress tensor  $\sigma(t)$  through the constitutive law of linearized elasticity  $\sigma(t) = \mathbb{C}e(t)$ . In a quasi-static setting, the equilibrium equations read as

$$\operatorname{div} \sigma(t) + f(t) \text{ in } \Omega, \quad \sigma(t)\nu = g(t) \text{ on } \Gamma_N,$$

where  $\nu$  denotes the outer unit normal to  $\partial\Omega$ . In plasticity, the stresses are constrained to remain below a yield stress at which permanent strains appear. Specifically, the deviatoric stress  $\sigma_D(t)$  must belong to a fixed compact and convex subset  $K$  of the deviatoric (trace free) matrices

$$\sigma_D(t) \in K.$$

If  $\sigma_D(t)$  lies inside the interior of  $K$ , the material behaves elastically ( $p(t) = 0$ ). On the other hand, if  $\sigma_D(t)$  reaches the boundary of  $K$  (called the yield surface), a plastic flow may develop, so that, after unloading, there will remain a non-trivial permanent plastic strain  $p(t)$ . Its evolution is described by the so-called flow rule

$$\dot{p}(t) \in N_K(\sigma_D(t))$$

where  $N_K(\sigma_D(t))$  is the normal cone to  $K$  at  $\sigma_D(t)$ . By arguments of convex analysis, the flow rule can be equivalently written as Hill's principle of maximum plastic work

$$\sigma_D(t) : \dot{p}(t) = \max_{\tau_D \in K} \tau_D : \dot{p}(t) =: H(\dot{p}(t)),$$

where  $H$  is the support function of  $K$ , and  $H(\dot{p}(t))$  identifies with the plastic dissipation.

In this self-contained note, we propose to show that rigid plasticity – that is the model where one formally sets  $\mathbb{C} = \infty$  (and correspondingly  $\dot{p}(t) = E\dot{u}(t)$ ,  $\operatorname{div} \dot{u}(t) = 0$ ) in the system above – can be derived as an asymptotic limit of small strain elasto-plasticity as  $\mathbb{C}$  actually gets larger and larger. Rigid-plastic models are particularly useful in order to compute analytical solutions in a plane-strain setting. Indeed, inside the plastic zone, the stress equations can be formally written

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as a non-linear hyperbolic system which is solved by the method of characteristics. The family of characteristics are the so-called *slip lines* along which some combinations of the stress remain constants, while the tangential velocities can jump. It thus seems appropriate to rigorously derive rigid-plasticity in order to investigate the hyperbolic structure of the equations. However, this later task falls outside the scope of the present work.

Notationwise, we denote by  $\mathbb{M}_{sym}^{n \times n}$  the set of symmetric  $n \times n$  matrices. If  $A$  and  $B \in \mathbb{M}_{sym}^{n \times n}$ , we use the Euclidean scalar product  $A : B := \text{tr}(AB)$  and the associated Euclidean norm  $|A| := \sqrt{A : A}$ . The subset  $\mathbb{M}_D^{n \times n}$  of  $\mathbb{M}_{sym}^{n \times n}$  stands for trace free symmetric matrices. If  $A \in \mathbb{M}_{sym}^{n \times n}$ , it can be orthogonally decomposed as

$$A = A_D + \frac{\text{tr } A}{n} I,$$

where  $A_D \in \mathbb{M}_D^{n \times n}$ , and  $I$  is the identity matrix in  $\mathbb{R}^n$ . The notation  $\odot$  stands for the symmetrized tensor product between vectors in  $\mathbb{R}^n$ , *i.e.*, if  $a$  and  $b \in \mathbb{R}^n$ ,  $(a \odot b)_{ij} = (a_i b_j + a_j b_i)/2$  for all  $1 \leq i, j \leq n$ . Note in particular that  $\frac{1}{\sqrt{2}}|a||b| \leq |a \odot b| \leq |a||b|$ .

The Lebesgue measure in  $\mathbb{R}^n$  and the  $(n-1)$ -dimensional Hausdorff measure are denoted by  $\mathcal{L}^n$  and  $\mathcal{H}^{n-1}$ , respectively. Given a locally compact set  $E \subset \mathbb{R}^n$  and a Euclidean space  $X$ , we denote by  $\mathcal{M}(E; X)$  (or simply  $\mathcal{M}(E)$  if  $X = \mathbb{R}$ ) the space of bounded Radon measures on  $E$  with values in  $X$ , endowed with the norm  $\|\mu\|_{\mathcal{M}(E; X)} := |\mu|(E)$ , where  $|\mu| \in \mathcal{M}(E)$  is the variation of the measure  $\mu$ . Moreover, if  $\nu$  is a non-negative Radon measure over  $E$ , we denote by  $d\mu/d\nu$  the Radon-Nikodym derivative of  $\mu$  with respect to  $\nu$ .

We use standard notation for Lebesgue and Sobolev spaces. In particular, for  $1 \leq p \leq \infty$ , the  $L^p$ -norms of the various quantities are denoted by  $\|\cdot\|_p$ . If  $U \subset \mathbb{R}^n$  is an open set, the space  $BD(U)$  of functions of bounded deformation in  $U$  is made of all functions  $u \in L^1(U; \mathbb{R}^n)$  such that  $Eu \in \mathcal{M}(U; \mathbb{M}_{sym}^{n \times n})$ , where  $Eu := (Du + Du^T)/2$  and  $Du$  is the distributional derivative of  $u$ . We refer to [14] for general properties of this space. Finally,  $H(\text{div}, U)$  stands for the Hilbert space of all  $\tau \in L^2(U; \mathbb{M}_{sym}^{n \times n})$  such that  $\text{div } \tau \in L^2(U; \mathbb{R}^n)$ .

## 2. THE ELASTO-PLASTIC MODEL

We now consider a homogeneous elasto-plastic material with Hooke's law given by a fourth order tensor  $\mathbb{C}$  satisfying the usual symmetry properties

$$\mathbb{C}_{ijkl} = \mathbb{C}_{jikl} = \mathbb{C}_{klij}, \quad \text{for all } 1 \leq i, j, k, l \leq n, \quad (2.1)$$

and the growth and coercivity assumptions

$$\alpha|\xi|^2 \leq \mathbb{C}\xi : \xi \leq \beta|\xi|^2, \quad \text{for all } \xi \in \mathbb{M}_{sym}^{n \times n}, \quad (2.2)$$

where  $\alpha$  and  $\beta > 0$ .

It occupies the domain  $\Omega$ , a bounded and connected open subset of  $\mathbb{R}^n$  with at least Lipschitz boundary (see Definition 2.1) and outer normal  $\nu$ . Its boundary  $\partial\Omega$  is split into the union of a Dirichlet part  $\Gamma_D$  which is non empty and open in the relative topology of  $\partial\Omega$ , a Neumann part  $\Gamma_N := \partial\Omega \setminus \overline{\Gamma_D}$ , and their common relative boundary denoted by  $\partial_{|\partial\Omega} \Gamma_D$ .

Standard plasticity is characterized by the fact that the deviatoric stress is constrained to stay in a fixed compact and convex subset  $K \subset \mathbb{M}_D^{n \times n}$  of deviatoric matrices. We further assume that

$$B(0, c_*) \subset K \subset B(0, c^*), \quad (2.3)$$

where  $0 < c_* < c^* < \infty$ , and denote by

$$\mathcal{K} := \{\sigma \in L^2(\Omega; \mathbb{M}_{sym}^{n \times n}) : \sigma_D(x) \in K \text{ for a.e. } x \in \Omega\}.$$

The support function of  $K$ , defined for any  $p \in \mathbb{M}_D^{n \times n}$  by  $H(p) := \sup_{\tau \in K} \tau : p$ , satisfies, according to (2.3),

$$c_*|p| \leq H(p) \leq c^*|p|, \quad \text{for all } p \in \mathbb{M}_{sym}^{n \times n}.$$

On the Dirichlet part  $\Gamma_D$  of the boundary, the body is subjected to a hard device, *i.e.*, a boundary displacement which is the trace on  $\Gamma_D$  of a function  $w \in AC([0, T]; H^1(\Omega; \mathbb{R}^n))$ . In addition, the body is subjected to two types of forces: bulk forces  $f \in AC([0, T]; L^n(\Omega; \mathbb{R}^n))$ , and surface forces  $g \in AC([0, T]; L^\infty(\Gamma_N; \mathbb{R}^n))$ , the latter acting on the Neumann part  $\Gamma_N$  of the boundary. It is classical to assume a uniform safe load condition (see [12]) which ensures the existence of a plastically, as well as statically admissible state of stress  $\pi$  associated with the pair  $(f, g)$ . Specifically, there exists  $\pi \in AC([0, T]; L^2(\Omega; \mathbb{M}_{sym}^{n \times n}))$  and some safety parameter  $c > 0$  such that

$$\begin{cases} \pi_D(t, x) + B(0, c) \subset K \text{ for a.e. } x \in \Omega \text{ and all } t \in [0, T] \\ \operatorname{div} \pi(t) + f(t) = 0 \text{ in } \Omega, \quad \pi(t)\nu = g(t) \text{ on } \Gamma_N. \end{cases}$$

Given a boundary datum  $\hat{w} \in H^1(\Omega; \mathbb{R}^n)$ , we define the space of all kinematically admissible triples as

$$\begin{aligned} \mathcal{A}(\hat{w}) := \{ (u, e, p) \in BD(\Omega) \times L^2(\Omega; \mathbb{M}_{sym}^{n \times n}) \times \mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n}) : \\ Eu = e + p \text{ in } \Omega, \quad p = (\hat{w} - u) \odot \nu \text{ on } \Gamma_D \}, \end{aligned}$$

where we still denote by  $u$  the trace of  $u$  on  $\partial\Omega$  (see [2]). We also define the space of all statically admissible stresses as

$$\Sigma := \{ \sigma \in L^2(\Omega; \mathbb{M}_{sym}^{n \times n}) : \operatorname{div} \sigma \in L^n(\Omega; \mathbb{R}^n), \quad \sigma\nu \in L^\infty(\Gamma_N; \mathbb{R}^n), \quad \sigma_D \in L^\infty(\Omega; \mathbb{M}_D^{n \times n}) \},$$

where  $\sigma\nu$  is the normal trace of  $\sigma \in H(\operatorname{div}, \Omega)$  which is well defined as an element of  $H^{-1/2}(\Gamma_N; \mathbb{R}^n)$ , the dual space of  $H_{00}^{1/2}(\Gamma_N; \mathbb{R}^n)$ .

Following [7, Section 6], we introduce the following class of domains for which a meaningful duality pairing between stresses and strains can be defined. Note that the class contains in particular  $\mathcal{C}^2$ -domains [10], as well as hypercubes where  $\Gamma_D$  is one of its faces [7, Section 6].

**Definition 2.1.** We say that  $\Omega$  is admissible if for any  $\sigma \in \Sigma$ , and any  $p \in \mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n})$ , with  $(u, e, p) \in \mathcal{A}(\hat{w})$  for some  $\hat{w} \in H^1(\Omega; \mathbb{R}^n)$ ,  $u \in BD(\Omega)$  and  $e \in L^2(\Omega; \mathbb{M}_{sym}^{n \times n})$ , the distribution defined for all  $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$  by

$$\begin{aligned} \langle [\sigma_D : p], \varphi \rangle := \int_{\Omega} \varphi \sigma : (E\hat{w} - e) dx - \int_{\Omega} \varphi \operatorname{div} \sigma \cdot (u - \hat{w}) dx \\ - \int_{\Omega} \sigma : [(u - \hat{w}) \odot \nabla \varphi] dx + \int_{\Gamma_N} \varphi \sigma \nu \cdot (u - \hat{w}) d\mathcal{H}^{n-1} \end{aligned}$$

extends to a bounded Radon measure in  $\mathbb{R}^n$  with  $||[\sigma_D : p]|| \leq \|\sigma_D\|_\infty |p|$ . In this case, its mass is given by

$$\langle \sigma_D, p \rangle := \langle [\sigma_D : p], 1 \rangle = \int_{\Omega} \sigma : (E\hat{w} - e) dx - \int_{\Omega} \operatorname{div} \sigma \cdot (u - \hat{w}) dx + \int_{\Gamma_N} \sigma \nu \cdot (u - \hat{w}) d\mathcal{H}^{n-1}. \quad (2.4)$$

For any  $e \in L^2(\Omega; \mathbb{M}_{sym}^{n \times n})$ , the elastic energy is

$$\mathcal{Q}(e) = \frac{1}{2} \int_{\Omega} \mathbb{C} e : e dx,$$

while, for any  $p \in \mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n})$ , the dissipation energy is the convex functional of measure (see [9, 6])

$$\mathcal{H}(p) := \int_{\Omega \cup \Gamma_D} H \left( \frac{dp}{d|p|} \right) d|p|.$$

If  $p : [0, T] \rightarrow \mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n})$ , we define the total dissipation between times  $a$  and  $b$  by

$$\mathcal{V}_{\mathcal{H}}(p; [a, b]) := \sup \left\{ \sum_{i=1}^N \mathcal{H}(p(t_i) - p^\varepsilon(t_{i-1})) : N \in \mathbb{N}, a = t_0 < t_1 < \dots < t_N = b \right\}.$$

If additionally  $p \in AC([0, T]; \mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n}))$ , then [4, Theorem 7.1] shows that

$$\mathcal{V}_{\mathcal{H}}(p; [a, b]) = \int_a^b \mathcal{H}(\dot{p}(s)) ds.$$

We finally impose the following initial condition on the evolution:  $(u_0, e_0, p_0) \in \mathcal{A}(w(0))$  with  $\sigma_0 := \mathbb{C}e_0$  such that

$$\operatorname{div} \sigma_0 + f(0) = 0 \text{ in } \Omega, \quad \sigma_0 \nu = g(0) \text{ on } \Gamma_N, \quad (\sigma_0)_D \in \mathcal{K}.$$

The following existence result has been established in [4, 7].

**Theorem 2.2.** *Under the previous assumptions, there exist a quasi-static evolution, i.e. a mapping  $t \mapsto (u(t), e(t), p(t))$  with the following properties*

$$u \in AC([0, T]; BD(\Omega)), \quad \sigma, e \in AC([0, T]; L^2(\Omega; \mathbb{M}_{sym}^{n \times n})), \quad p \in AC([0, T]; \mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n})),$$

$$(u(0), e(0), p(0)) = (u_0, e_0, p_0),$$

and for all  $t \in [0, T]$ ,

$$\begin{cases} Eu(t) = e(t) + p(t) \text{ in } \Omega, \\ p(t) = (w(t) - u(t)) \odot \nu \text{ on } \Gamma_D, \\ \sigma(t) = \mathbb{C}e(t) \text{ in } \Omega, \end{cases}$$

$$\begin{cases} \operatorname{div} \sigma(t) + f(t) = 0 \text{ in } \Omega, \\ \sigma(t) \nu = g(t) \text{ on } \Gamma_N, \\ \sigma_D(t) \in \mathcal{K}, \end{cases}$$

and for a.e.  $t \in [0, T]$ ,

$$H(\dot{p}(t)) = [\sigma_D(t) : \dot{p}(t)] \text{ in } \mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n}). \quad (2.5)$$

**Remark 2.3.** Equation (2.5) is a measure-theoretic formulation of the usual flow rule of perfect plasticity. Using the definition (2.4) of duality, it can be equivalently written as an energy balance

$$\begin{aligned} \mathcal{Q}(e(t)) + \int_0^t \mathcal{H}(\dot{p}(s)) ds &= \mathcal{Q}(e_0) + \int_0^t \int_{\Omega} \sigma(s) : E\dot{w}(s) dx ds \\ &\quad + \int_0^t \int_{\Omega} f(s) \cdot (\dot{u}(s) - \dot{w}(s)) dx ds + \int_0^t \int_{\Gamma_N} g(s) \cdot (\dot{u}(s) - \dot{w}(s)) d\mathcal{H}^{n-1} ds, \end{aligned}$$

or equivalently, according to the safe-load condition,

$$\begin{aligned} \mathcal{Q}(e(t)) + \int_0^t \mathcal{H}(\dot{p}(s)) ds &- \int_0^t \langle \pi_D(s), \dot{p}(s) \rangle ds + \int_{\Omega} \pi(t) : (Ew(t) - e(t)) dx \\ &= \mathcal{Q}(e_0) + \int_{\Omega} \pi(0) : (Ew(0) - e_0) dx + \int_0^t \int_{\Omega} \sigma(s) : E\dot{w}(s) dx ds \\ &\quad + \int_0^t \int_{\Omega} \dot{\pi}(s) : (Ew(s) - e(s)) dx ds. \quad (2.6) \end{aligned}$$

## 3. THE RIGID-PLASTIC MODEL

In order to derive the rigid-plastic model from elasto-plasticity, we assume that

$$\mathbb{C}^\varepsilon = \varepsilon^{-1} \mathbb{C}, \quad \text{where } \mathbb{C} \text{ satisfies (2.1) and (2.2),} \quad (3.1)$$

and  $\varepsilon \rightarrow 0^+$ . In addition, we suppose that the boundary data are compatible with rigid plasticity, that is

$$\operatorname{div} w(t) = 0 \text{ in } \Omega, \quad (3.2)$$

and, for simplicity, that the initial data satisfy

$$e_0 = \sigma_0 = 0. \quad (3.3)$$

**Theorem 3.1.** *Let  $u^\varepsilon$ ,  $e^\varepsilon$ ,  $p^\varepsilon$  and  $\sigma^\varepsilon$  be the solutions given by Theorem 2.2. There exist a subsequence (not relabeled), and functions  $u \in AC([0, T]; BD(\Omega))$  and  $\sigma \in L^2(0, T; L^2(\Omega; \mathbb{M}_{sym}^{n \times n}))$  such that*

$$\begin{aligned} u^\varepsilon(t) &\rightharpoonup u(t) \text{ weakly}^* \text{ in } BD(\Omega), \text{ for all } t \in [0, T], \\ \sigma^\varepsilon &\rightharpoonup \sigma \text{ weakly in } L^2(0, T; L^2(\Omega; \mathbb{M}_{sym}^{n \times n})). \end{aligned}$$

Denoting by  $v := \dot{u} \in L_{w*}^\infty(0, T; BD(\Omega))$ , then for a.e.  $t \in [0, T]$ , we have

$$\begin{cases} -\operatorname{div} \sigma(t) = f(t) \text{ in } \Omega, \\ \sigma(t)\nu = g(t) \text{ on } \Gamma_N, \\ \sigma(t) \in \mathcal{K}, \end{cases} \quad \begin{cases} \operatorname{div} v(t) = 0 \text{ in } \Omega, \\ (\dot{w}(t) - v(t)) \cdot \nu = 0 \text{ on } \Gamma_D, \\ H(Ev(t)) = [\sigma_D(t) : Ev(t)] \text{ in } \Omega \cup \Gamma_D. \end{cases} \quad (3.4)$$

The remaining of this paper is devoted to the proof of Theorem 3.1.

**Remark 3.2.** Although  $Eu(t)$  is a measure *a priori* defined in  $\Omega$ , we tacitly extend it by  $(w(t) - u(t)) \odot \nu$  on  $\Gamma_D$  so that  $Eu(t) \in \mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n})$ .

**Remark 3.3.** In contrast with the framework of classical elasto-plasticity, that of rigid plasticity only involves the velocity field, and not the displacement field itself. As expressed above, time is merely a parameter, although the associated measurability properties of the various fields are obtained through the limit process  $\varepsilon \searrow 0$  and would be difficult to obtain directly from the limit formulation.

**3.1. A priori estimates.** In this section all constants are independent of  $\varepsilon$ . We start with an estimate of the stress. Since  $\sigma_D^\varepsilon(t) \in K$  in  $\Omega$ , and  $K$  is bounded by (2.3), we first deduce that

$$\sup_{t \in [0, T]} \|\sigma_D^\varepsilon(t)\|_\infty \leq C. \quad (3.5)$$

The following result allows us to bound the hydrostatic stress.

**Lemma 3.4.** *There exists a bounded sequence  $(c^\varepsilon)_{\varepsilon > 0}$  in  $L^2(0, T)$  such that for each  $\varepsilon > 0$ ,*

$$\int_0^T \left\| \frac{\operatorname{tr} \sigma^\varepsilon(t)}{n} + c^\varepsilon(t) \right\|_2^2 dt \leq C.$$

*Proof.* Since the mapping  $t \mapsto \sigma^\varepsilon(t)$  belongs to  $L^2(0, T; H(\operatorname{div}, \Omega))$ , there is a sequence  $(\sigma_k^\varepsilon)_{k \in \mathbb{N}}$  of  $H(\operatorname{div}, \Omega)$ -valued simple functions such that  $\sigma_k^\varepsilon \rightarrow \sigma^\varepsilon$  strongly in  $L^2(0, T; H(\operatorname{div}, \Omega))$  as  $k \rightarrow +\infty$ . For all  $k \in \mathbb{N}$  and all  $t \in [0, T]$ , we have

$$\nabla \left( \frac{\operatorname{tr} \sigma_k^\varepsilon(t)}{n} \right) = \operatorname{div} \sigma_k^\varepsilon(t) - \operatorname{div}(\sigma_k^\varepsilon)_D(t) \text{ in } \Omega$$

which leads to

$$\int_0^T \left\| \nabla \left( \frac{\operatorname{tr} \sigma_k^\varepsilon(t)}{n} \right) \right\|_{H^{-1}(\Omega; \mathbb{R}^n)}^2 dt \leq \int_0^T \|\operatorname{div} \sigma_k^\varepsilon(t)\|_{H^{-1}(\Omega; \mathbb{R}^n)}^2 dt + \int_0^T \|(\sigma_k^\varepsilon)_D(t)\|_2^2 dt.$$

Since  $\operatorname{div} \sigma_k^\varepsilon \rightarrow \operatorname{div} \sigma^\varepsilon$  in  $L^2(0, T; L^2(\Omega; \mathbb{R}^n))$  and  $-\operatorname{div} \sigma^\varepsilon = f \in L^2(0, T; L^2(\Omega; \mathbb{R}^n))$ , we deduce that the first integral in the right-hand-side of the previous inequality is uniformly bounded with respect to  $\varepsilon$  and  $k$ . The second integral is bounded as well since  $(\sigma_k^\varepsilon)_D \rightarrow \sigma_D^\varepsilon$  in  $L^2(0, T; L^2(\Omega; \mathbb{M}_D^{n \times n}))$ , and  $(\sigma_D^\varepsilon)_{\varepsilon > 0}$  is uniformly bounded in that space in view of (3.5). Consequently, there exists a constant  $C > 0$  (independent of  $k$  and  $\varepsilon$ ) such that

$$\int_0^T \left\| \nabla \left( \frac{\operatorname{tr} \sigma_k^\varepsilon(t)}{n} \right) \right\|_{H^{-1}(\Omega; \mathbb{R}^n)}^2 dt \leq C.$$

Next, according to [8, Corollary 2.1] (see also [13, Lemma 9] in the case of smooth boundaries), for each  $\varepsilon > 0$ ,  $k \in \mathbb{N}$  and  $t \in [0, T]$ , there exists some  $c_k^\varepsilon(t) \in \mathbb{R}$  such that

$$\left\| \frac{\operatorname{tr} \sigma_k^\varepsilon(t)}{n} + c_k^\varepsilon(t) \right\|_2 \leq C_\Omega \left\| \nabla \left( \frac{\operatorname{tr} \sigma_k^\varepsilon(t)}{n} \right) \right\|_{H^{-1}(\Omega; \mathbb{R}^n)},$$

for some constant  $C_\Omega > 0$  only depending on  $\Omega$ . Note that, since the mapping  $t \mapsto \operatorname{tr} \sigma_k^\varepsilon(t)$  is a simple  $L^2(\Omega)$ -valued function,  $t \mapsto c_k^\varepsilon(t)$  is a simple real-valued measurable function as well. Additionally,

$$\int_0^T \left\| \frac{\operatorname{tr} \sigma_k^\varepsilon(t)}{n} + c_k^\varepsilon(t) \right\|_2^2 dt \leq C, \quad (3.6)$$

where  $C > 0$  is again independent of  $k$  and  $\varepsilon$ . Setting  $\hat{\sigma}_k^\varepsilon := \sigma_k^\varepsilon + c_k^\varepsilon I$  yields

$$\int_0^T \|\hat{\sigma}_k^\varepsilon(t)\|_{H(\operatorname{div}, \Omega)}^2 dt \leq C,$$

and thus,

$$\int_0^T \|\hat{\sigma}_k^\varepsilon(t)\nu\|_{H^{-1/2}(\Gamma_N; \mathbb{R}^n)}^2 dt \leq C.$$

Using that  $\sigma_k^\varepsilon \nu \rightarrow \sigma^\varepsilon \nu = g$  in  $L^2(0, T; H^{-1/2}(\Gamma_N; \mathbb{R}^n))$  and that  $g \in L^2(0, T; L^2(\Gamma_N; \mathbb{R}^n))$ , we obtain

$$\begin{aligned} & \int_0^T |c_k^\varepsilon(t)|^2 dt \|\nu\|_{H^{-1/2}(\Gamma_N; \mathbb{R}^n)}^2 \\ & \leq \int_0^T \|\hat{\sigma}_k^\varepsilon(t)\nu\|_{H^{-1/2}(\Gamma_N; \mathbb{R}^n)}^2 dt + \int_0^T \|\sigma_k^\varepsilon(t)\nu\|_{H^{-1/2}(\Gamma_N; \mathbb{R}^n)}^2 dt \leq C, \end{aligned} \quad (3.7)$$

for some constant  $C > 0$ , independent of  $k$  and  $\varepsilon$ . Therefore, the sequence  $(c_k^\varepsilon)_{k \in \mathbb{N}}$  is bounded in  $L^2(0, T)$  and a subsequence converges weakly in that space to some  $c^\varepsilon \in L^2(0, T)$ . Passing to the lower limit in (3.6) implies that

$$\int_0^T \left\| \frac{\operatorname{tr} \sigma^\varepsilon(t)}{n} + c^\varepsilon(t) \right\|_2^2 dt \leq C,$$

while (3.7) shows that  $(c^\varepsilon)_{\varepsilon > 0}$  is bounded in  $L^2(0, T)$ . □

As a consequence of the previous result and of (3.5), we deduce that

$$\int_0^T \|\sigma^\varepsilon(t)\|_2^2 dt \leq C. \quad (3.8)$$

Next, according to the energy balance (2.6), [4, Lemma 3.2], assumptions (3.2)–(3.3), and Cauchy-Schwarz inequality, we infer that

$$\begin{aligned} \frac{1}{2} \int_{\Omega} \mathbb{C}^\varepsilon e^\varepsilon(t) : e^\varepsilon(t) dx &\leq \int_{\Omega} \pi(t) : (e^\varepsilon(t) - Ew(t)) dx + \int_{\Omega} \pi(0) : Ew(0) dx \\ &\quad + \int_0^t \int_{\Omega} \sigma_D^\varepsilon(s) : E\dot{w}(s) dx ds + \int_0^t \int_{\Omega} \dot{\pi}(s) : (Ew(s) - e^\varepsilon(s)) dx ds \\ &\leq C \left( \sup_{t \in [0, T]} \|\pi(t)\|_2 + \int_0^T \|\dot{\pi}(s)\|_2 ds \right) \left( \sup_{t \in [0, T]} \|e^\varepsilon(t)\|_2 + \sup_{t \in [0, T]} \|Ew(t)\|_2 \right) \\ &\quad + \sup_{t \in [0, T]} \|\sigma_D^\varepsilon(t)\|_\infty \int_0^T \|E\dot{w}(s)\|_2 ds, \end{aligned}$$

which implies, according to the assumption (3.1) on  $\mathbb{C}^\varepsilon$  together with Young's inequality, that

$$\sup_{t \in [0, T]} \|e^\varepsilon(t)\|_2 \leq C\sqrt{\varepsilon}. \quad (3.9)$$

Using again the energy balance (2.6), Cauchy-Schwarz inequality and (3.9), we find that

$$\begin{aligned} \int_0^t \mathcal{H}(\dot{p}^\varepsilon(s)) ds - \int_0^t \langle \pi_D(s), \dot{p}^\varepsilon(s) \rangle ds &\leq \int_{\Omega} \pi(t) : (e^\varepsilon(t) - Ew(t)) dx + \int_{\Omega} \pi(0) : Ew(0) dx \\ &\quad + \int_0^t \int_{\Omega} \sigma_D^\varepsilon(s) : E\dot{w}(s) dx ds + \int_0^t \int_{\Omega} \dot{\pi}(s) : (Ew(s) - e^\varepsilon(s)) dx ds \leq C. \end{aligned}$$

Applying [4, Lemma 3.2] again yields

$$\int_0^T \|\dot{p}^\varepsilon(s)\|_{\mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n})} ds \leq C, \quad (3.10)$$

and thus

$$\sup_{t \in [0, T]} \|p^\varepsilon(t)\|_{\mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n})} \leq C. \quad (3.11)$$

For the displacement, Poincaré-Korn's inequality (see [14, Chap. 2, Rmk. 2.5(ii)]) yields

$$\begin{aligned} \|u^\varepsilon(t)\|_{BD(\Omega)} &\leq c \left( \int_{\Gamma_D} |u^\varepsilon(t)| d\mathcal{H}^{n-1} + \|Eu^\varepsilon(t)\|_{\mathcal{M}(\Omega; \mathbb{M}_{sym}^{n \times n})} \right) \\ &\leq c \left( \int_{\Gamma_D} |w(t)| d\mathcal{H}^{n-1} + \int_{\Gamma_D} |u^\varepsilon(t) - w(t)| d\mathcal{H}^{n-1} + \|Eu^\varepsilon(t)\|_{\mathcal{M}(\Omega; \mathbb{M}_{sym}^{n \times n})} \right) \\ &\leq c \left( \|w(t)\|_{L^1(\Gamma_D; \mathbb{R}^n)} + \|p^\varepsilon(t)\|_{\mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n})} + \|e^\varepsilon(t)\|_2 \right) \leq C, \end{aligned} \quad (3.12)$$

where we have used (3.9) and (3.11) in the last inequality.

**3.2. Convergences.** According to the stress estimate (3.8), there exist a subsequence (not relabeled) and  $\sigma \in L^2(0, T; L^2(\Omega; \mathbb{M}_{sym}^{n \times n}))$  such that

$$\sigma^\varepsilon \rightharpoonup \sigma \text{ weakly in } L^2(0, T; L^2(\Omega; \mathbb{M}_{sym}^{n \times n})). \quad (3.13)$$

Consequently, since for all  $t \in [0, T]$ , we have  $-\operatorname{div} \sigma^\varepsilon(t) = f(t)$  in  $\Omega$  and  $\sigma^\varepsilon(t)\nu = g(t)$  on  $\Gamma_N$ , we infer that for a.e.  $t \in [0, T]$ ,

$$-\operatorname{div} \sigma(t) = f(t) \text{ in } \Omega, \quad \sigma(t)\nu = g(t) \text{ on } \Gamma_N.$$

In addition, since  $\sigma_D^\varepsilon(t) \in \mathcal{K}$  for all  $t \in [0, T]$ , then

$$\sigma_D(t) \in \mathcal{K} \text{ for a.e. } t \in [0, T].$$

We then apply Helly's selection principle (see [11, Theorem 3.2]) which ensures, thanks to (3.10), the existence of a further subsequence (independent of time and still not relabeled) such that

$$p^\varepsilon(t) \rightharpoonup p(t) \text{ weakly}^* \text{ in } \mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n}), \text{ for all } t \in [0, T], \quad (3.14)$$

for some  $p \in BV([0, T]; \mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n}))$ .

Next according to (3.9), we have that

$$e^\varepsilon \rightarrow 0 \text{ strongly in } L^\infty(0, T; L^2(\Omega; \mathbb{M}_{sym}^{n \times n})). \quad (3.15)$$

Finally, as a consequence of the displacement estimate (3.12), for each  $t \in [0, T]$ , there exists a further subsequence  $(u^{\varepsilon_j}(t))_{j \in \mathbb{N}}$  (now possibly depending on  $t$ ) such that  $u^{\varepsilon_j}(t) \rightharpoonup u(t)$  weakly\* in  $BD(\Omega)$ , for some  $u(t) \in BD(\Omega)$ . Note that by (3.14)–(3.15), for a.e.  $t \in [0, T]$ , one has  $Eu(t) = p(t)$  in  $\Omega$  and, by [4, Lemma 2.1],  $p(t) = (w(t) - u(t)) \odot \nu$  on  $\Gamma_D$  which shows that  $u(t)$  is uniquely determined, and thus that the full sequence

$$u^\varepsilon(t) \rightharpoonup u(t) \text{ weakly}^* \text{ in } BD(\Omega), \text{ for all } t \in [0, T]. \quad (3.16)$$

In particular, since  $Eu(t) = p(t) \in \mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n})$ , we also deduce that

$$\operatorname{div} u(t) = 0 \text{ in } \Omega, \quad (w(t) - u(t)) \cdot \nu = 0 \text{ on } \Gamma_D. \quad (3.17)$$

**3.3. Flow rule.** According to the energy balance (2.6) and the fact that the plastic strain  $p^\varepsilon \in AC([0, T]; \mathcal{M}(\overline{\Omega}; \mathbb{M}_D^{n \times n}))$ , we can integrate by parts in time, so that for all  $t \in [0, T]$ ,

$$\begin{aligned} \mathcal{V}_\mathcal{H}(p^\varepsilon; [0, t]) &+ \int_\Omega \pi(t) : (Ew(t) - e^\varepsilon(t)) \, dx - \langle \pi_D(t), p^\varepsilon(t) \rangle \\ &\leq \int_\Omega \pi(0) : Ew(0) \, dx - \langle \pi_D(0), p_0 \rangle + \int_0^t \int_\Omega \sigma_D^\varepsilon(s) : E\dot{w}(s) \, dx \, ds \\ &\quad + \int_0^t \int_\Omega \dot{\pi}(s) : (Ew(s) - e^\varepsilon(s)) \, dx \, ds - \int_0^t \langle \dot{\pi}_D(s), p^\varepsilon(s) \rangle \, ds. \end{aligned}$$

Since by (3.14)–(3.16)  $p^\varepsilon(t) \rightharpoonup Eu(t)$  weakly\* in  $\mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n})$  for a.e.  $t \in [0, T]$ , Reshetnyak lower semicontinuity theorem, (3.13), (3.15), (3.16) and the definition (2.4) of duality ensures that

$$\begin{aligned} \mathcal{V}_\mathcal{H}(Eu; [0, t]) &+ \int_\Omega \pi(t) : Ew(t) \, dx - \langle \pi_D(t), Eu(t) \rangle \\ &\leq \int_\Omega \pi(0) : Ew(0) \, dx - \langle \pi_D(0), Eu_0 \rangle + \int_0^t \int_\Omega \sigma_D(s) : E\dot{w}(s) \, dx \, ds \\ &\quad + \int_0^t \int_\Omega \dot{\pi}(s) : Ew(s) \, dx \, ds - \int_0^t \langle \dot{\pi}_D(s), Eu(s) \rangle \, ds. \quad (3.18) \end{aligned}$$

We now show the converse inequality. Since  $\sigma_D \in L^1(0, T; L^2(\Omega; \mathbb{M}_D^{n \times n}))$ , while  $u - w \in L^1(0, T; L^{\frac{n}{n-1}}(\Omega; \mathbb{R}^n))$ , and  $u - w \in L^1(0, T; L^1(\Gamma_N; \mathbb{R}^n))$ , [5, Lemma 7.5] implies the existence of a subdivision  $0 = t_0 < t_1 < \dots < t_k = t$  of the time interval  $[0, t]$  such that

$$\sum_{i=1}^k \chi_{[t_{i-1}, t_i]}(\sigma_D(t_i), u(t_i) - w(t_i), u(t_i) - w(t_i)) \rightarrow (\sigma_D, u - w, u - w)$$

and

$$\sum_{i=1}^k \chi_{[t_{i-1}, t_i]}(\sigma_D(t_{i-1}), u(t_{i-1}) - w(t_{i-1}), u(t_{i-1}) - w(t_{i-1})) \rightarrow (\sigma_D, u - w, u - w)$$



strongly in  $L^1(0, T; L^2(\Omega; \mathbb{M}_D^{n \times n})) \times L^1(0, T; L^{\frac{n}{n-1}}(\Omega; \mathbb{R}^n)) \times L^1(0, T; L^1(\Gamma_N; \mathbb{R}^n))$ , as  $\max_{1 \leq i \leq k} (t_i - t_{i-1}) \rightarrow 0$ . According to Proposition 3.9 in [7] and to the fact that  $\Omega$  is admissible, we infer that for each  $1 \leq i \leq k$ ,

$$\begin{aligned} \mathcal{H}(Eu(t_i) - Eu(t_{i-1})) &\geq \langle \sigma_D(t_i), Eu(t_i) - Eu(t_{i-1}) \rangle \\ &= \int_{\Omega} \sigma_D(t_i) : (Ew(t_i) - Ew(t_{i-1})) dx + \int_{\Omega} f(t_i) \cdot (u(t_i) - u(t_{i-1}) - w(t_i) + w(t_{i-1})) dx \\ &\quad + \int_{\Gamma_N} g(t_i) \cdot (u(t_i) - u(t_{i-1}) - w(t_i) + w(t_{i-1})) d\mathcal{H}^{n-1}. \end{aligned}$$

Summing up for  $i = 1, \dots, k$ , and performing discrete integration by parts yields

$$\begin{aligned} \mathcal{V}_{\mathcal{H}}(Eu, [0, t]) &\geq \sum_{i=1}^k \int_{t_{i-1}}^{t_i} \int_{\Omega} \sigma_D(t_i) : E\dot{w}(s) dx ds \\ &\quad - \sum_{i=1}^{k-1} \int_{t_i}^{t_{i+1}} \int_{\Omega} \dot{f}(s) \cdot (u(t_i) - w(t_i)) dx ds - \sum_{i=1}^{k-1} \int_{t_i}^{t_{i+1}} \int_{\Gamma_N} \dot{g}(s) \cdot (u(t_i) - w(t_i)) d\mathcal{H}^{n-1} ds \\ &\quad + \int_{\Omega} f(t) \cdot (u(t) - w(t)) dx + \int_{\Gamma_N} g(t) \cdot (u(t) - w(t)) d\mathcal{H}^{n-1} \\ &\quad - \int_{\Omega} f(t_1) \cdot (u_0 - w(0)) dx - \int_{\Gamma_N} g(t_1) \cdot (u_0 - w(0)) d\mathcal{H}^{n-1}. \end{aligned}$$

Passing to the limit as  $\max_{1 \leq i \leq k} (t_i - t_{i-1}) \rightarrow 0$ , and invoking the dominated convergence theorem yields

$$\begin{aligned} \mathcal{V}_{\mathcal{H}}(Eu, [0, t]) &\geq \int_0^t \int_{\Omega} \sigma_D(s) : E\dot{w}(s) dx ds \\ &\quad - \int_0^t \int_{\Omega} \dot{f}(s) \cdot (u(s) - w(s)) dx ds - \int_0^t \int_{\Gamma_N} \dot{g}(s) \cdot (u(s) - w(s)) d\mathcal{H}^{n-1} ds \\ &\quad + \int_{\Omega} f(t) \cdot (u(t) - w(t)) dx + \int_{\Gamma_N} g(t) \cdot (u(t) - w(t)) d\mathcal{H}^{n-1} \\ &\quad - \int_{\Omega} f(0) \cdot (u_0 - w(0)) dx - \int_{\Gamma_N} g(0) \cdot (u_0 - w(0)) d\mathcal{H}^{n-1}, \end{aligned}$$

and using the definition (2.4) of duality

$$\begin{aligned} \mathcal{V}_{\mathcal{H}}(Eu; [0, t]) &+ \int_{\Omega} \pi(t) : Ew(t) dx - \langle \pi_D(t), Eu(t) \rangle \\ &\geq \int_{\Omega} \pi(0) : Ew(0) dx - \langle \pi_D(0), Eu_0 \rangle + \int_0^t \int_{\Omega} \sigma_D(s) : E\dot{w}(s) dx ds \\ &\quad + \int_0^t \int_{\Omega} \dot{\pi}(s) : Ew(s) dx ds - \int_0^t \langle \dot{\pi}_D(s), Eu(s) \rangle ds. \end{aligned}$$

Thus, combining with (3.18) leads to the equality in the previous inequality, or still, integrating by parts with respect to time

$$\begin{aligned} \mathcal{V}_{\mathcal{H}}(Eu; [0, t]) &= \langle \pi_D(t), Eu(t) \rangle - \langle \pi_D(0), Eu_0 \rangle \\ &\quad + \int_0^t \int_{\Omega} (\sigma_D(s) - \pi_D(s)) : E\dot{w}(s) dx ds - \int_0^t \langle \dot{\pi}_D(s), Eu(s) \rangle ds. \quad (3.19) \end{aligned}$$

According to [4, Lemma 3.2], for all  $0 \leq t_1 \leq t_2 \leq T$ ,

$$\begin{aligned} c\|Eu(t_2) - Eu(t_1)\|_{\mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n})} &\leq \mathcal{H}(Eu(t_2) - Eu(t_1)) - \langle \pi_D(t_2), Eu(t_2) - Eu(t_1) \rangle \\ &\leq \mathcal{V}_{\mathcal{H}}(Eu, [t_1, t_2]) - \langle \pi_D(t_2), Eu(t_2) - Eu(t_1) \rangle. \end{aligned}$$

In view of (3.19), we get that

$$\begin{aligned} c\|Eu(t_2) - Eu(t_1)\|_{\mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n})} &\leq \langle \pi_D(t_2) - \pi_D(t_1), Eu(t_1) \rangle \\ &\quad + \int_{t_1}^{t_2} \int_{\Omega} (\sigma_D(s) - \pi_D(s)) : E\dot{u}(s) \, dx \, ds - \int_{t_1}^{t_2} \langle \dot{\pi}_D(s), Eu(s) \rangle \, ds. \end{aligned}$$

Since  $Eu = p$  and  $p \in BV([0, T]; \mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n}))$ , we get that  $Eu \in L_{w*}^{\infty}(0, T; \mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n}))$ , and thus

$$\begin{aligned} c\|Eu(t_2) - Eu(t_1)\|_{\mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n})} &\leq \int_{t_1}^{t_2} \left\{ \|Eu(t_1)\|_{\mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n})} \|\dot{\pi}_D(s)\|_{\infty} \right. \\ &\quad \left. + (\|\pi_D(s)\|_2 + \|\sigma_D(s)\|_2) \|E\dot{u}(s)\|_2 + \|\dot{\pi}_D(s)\|_{\infty} \|Eu(s)\|_{\mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n})} \right\} ds. \end{aligned}$$

The integrand being sommable, it ensures that the strain  $Eu \in AC([0, T]; \mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n}))$ , and by the Poincaré-Korn inequality that  $u \in AC([0, T]; BD(\Omega))$ . Thus, integrating by part with respect to time and space in the energy equality (3.19),

$$\begin{aligned} \int_0^t \mathcal{H}(E\dot{u}(s)) \, ds &= \mathcal{V}_{\mathcal{H}}(Eu, [0, t]) = \int_0^t \int_{\Omega} \sigma_D(s) : E\dot{u}(s) \, dx \, ds \\ &\quad + \int_0^t \int_{\Omega} f(s) \cdot (\dot{u}(s) - \dot{w}(s)) \, dx \, ds + \int_0^t \int_{\Gamma_N} g(s) \cdot (\dot{u}(s) - \dot{w}(s)) \, d\mathcal{H}^{n-1} \, ds, \end{aligned}$$

and deriving this equality with respect to time yields, thanks to (2.4), for a.e.  $t \in [0, T]$ ,

$$\mathcal{H}(E\dot{u}(t)) = \langle \sigma_D(t), E\dot{u}(t) \rangle.$$

Since, by [7, Proposition 3.9],  $H(E\dot{u}(t)) \geq [\sigma_D(t) : E\dot{u}(t)]$  in  $\mathcal{M}(\Omega \cup \Gamma_D)$ , we finally deduce that  $H(E\dot{u}(t)) = [\sigma_D(t) : E\dot{u}(t)]$  in  $\mathcal{M}(\Omega \cup \Gamma_D)$ .

Denoting by  $v = \dot{u}$  the velocity, we proved that  $v \in L_{w*}^{\infty}(0, T; BD(\Omega))$ , and recalling (3.17), we have for a.e.  $t \in [0, T]$ ,

$$\operatorname{div} v(t) = 0 \text{ in } \Omega, \quad (\dot{w}(t) - v(t)) \cdot \nu = 0 \text{ on } \Gamma_D,$$

and

$$H(Ev(t)) = [\sigma_D(t) : Ev(t)] \text{ in } \Omega \cup \Gamma_D.$$

#### 4. UNIQUENESS AND REGULARITY ISSUES FOR THE STRESS WITH A VON MISES YIELD CRITERION

We now specialize to the case where  $K := \{\tau_D \in \mathbb{M}_D^{n \times n} : |\tau_D| \leq 1\}$ . In such a setting, it is known (see [3]) when elasto-plasticity is considered the stress field is unique and belongs to  $H_{\text{loc}}^1(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$ . These properties fail in the case of rigid-plasticity as demonstrated below.

**Example 4.1.** Let us consider a two-dimensional body occupying the square  $\Omega = (0, 1)^2$  in its reference configuration (the generalization to the  $n$ -dimensional case is obvious). We also assume that the boundary conditions are of pure Dirichlet type with a rigid body motion  $\dot{w}(x) = Ax + b$  (where  $A \in \mathbb{M}^{n \times n}$  is such that  $A^T = -A^T$ , and  $b \in \mathbb{R}^n$ ) as boundary datum.

Then, defining  $v(x) = Ax + b$  for all  $x \in \Omega$  ensures that  $Ev = 0$  in  $\Omega$ . In particular, all equations on  $v$  are satisfied. Now define the stress as

$$\sigma(x) = \begin{pmatrix} f(x_2) & c \\ c & g(x_1) \end{pmatrix}$$

where  $c \in \mathbb{R}$ ,  $f, g \in L^\infty(0, 1)$  so that  $\operatorname{div} \sigma = 0$  in  $\Omega$ . In particular

$$\sigma_D(x) = \begin{pmatrix} \frac{f(x_2) - g(x_1)}{2} & c \\ c & \frac{g(x_1) - f(x_2)}{2} \end{pmatrix}$$

and  $|\sigma_D(x)|^2 \leq 2c^2 + |f(x_2)|^2 + |g(x_1)|^2$  for a.e.  $x \in \Omega$ . Assuming that  $\sqrt{2c^2 + \|f\|_\infty^2 + \|g\|_\infty^2} < 1/2$ , we deduce that the one parameter family  $\sigma^\lambda := \lambda \sigma$  still satisfies  $\operatorname{div} \sigma^\lambda = 0$  and  $|\sigma_D^\lambda| < 1$  in  $\Omega$  provided that  $|\lambda| \leq 2$ .

In general, a certain amount of uniqueness holds true as shown below. It uses a notion of precise representative for the stress field first introduced in [1] (see also [4]).

**Proposition 4.2.** *Let  $(\sigma^1, v^1), (\sigma^2, v^2) \in L^2(\Omega; \mathbb{M}_{sym}^{n \times n}) \times BD(\Omega)$  be two solutions of the rigid-plastic model (3.4) at a given time  $t = t_0$ . Then,*

- *There exist two  $|Ev^1|$ -measurable functions  $\hat{\sigma}_D^1$  and  $\hat{\sigma}_D^2 \in L^\infty_{|Ev^1|}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n})$  such that  $\hat{\sigma}_D^1 = \sigma^1$  and  $\hat{\sigma}_D^2 = \sigma_D^2$   $\mathcal{L}^n$ -a.e. in  $\Omega \cup \Gamma_D$ , and*

$$\hat{\sigma}_D^1 = \hat{\sigma}_D^2 \quad |Ev^1| \text{-a.e. in } \Omega \cup \Gamma_D;$$

- *There exist two  $|Ev^2|$ -measurable functions  $\tilde{\sigma}_D^1$  and  $\tilde{\sigma}_D^2 \in L^\infty_{|Ev^2|}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n})$  such that  $\tilde{\sigma}_D^1 = \sigma^1$  and  $\tilde{\sigma}_D^2 = \sigma_D^2$   $\mathcal{L}^n$ -a.e. in  $\Omega \cup \Gamma_D$ , and*

$$\tilde{\sigma}_D^1 = \tilde{\sigma}_D^2 \quad |Ev^2| \text{-a.e. in } \Omega \cup \Gamma_D.$$

*Proof.* Since  $(\sigma^1, v^1), (\sigma^2, v^2)$  are two solutions of the rigid-plastic model (3.4), the following inequalities in  $\mathcal{M}(\Omega \cup \Gamma_D)$  hold true

$$[\sigma_D^1 : Ev^1] = |Ev^1| \geq [\sigma_D^2 : Ev^1], \quad [\sigma_D^2 : Ev^2] = |Ev^2| \geq [\sigma_D^1 : Ev^2].$$

As a consequence,

$$[(\sigma_D^1 - \sigma_D^2) : Ev^1] \geq 0, \quad [(\sigma_D^2 - \sigma_D^1) : Ev^2] \geq 0,$$

and thus,

$$[(\sigma_D^1 - \sigma_D^2) : (Ev^1 - Ev^2)] \geq 0.$$

In addition, by definition (2.4) of the duality pairing, the total mass of the measure on the left-hand side of the previous inequality is given by

$$\langle \sigma_D^1 - \sigma_D^2, Ev^1 - Ev^2 \rangle = 0.$$

It thus follows that

$$[(\sigma_D^1 - \sigma_D^2) : Ev^1] = 0, \quad [(\sigma_D^2 - \sigma_D^1) : Ev^2] = 0,$$

or still that

$$[\sigma_D^1 : Ev^1] = |Ev^1| = [\sigma_D^2 : Ev^1], \quad [\sigma_D^2 : Ev^2] = |Ev^2| = [\sigma_D^1 : Ev^2]. \quad (4.1)$$

Arguing as in [4], since  $\mathcal{L}^n$  and  $E^s v^1$  are mutually singular Borel measures, it is possible to find two disjoint Borel sets  $A$  and  $B \subset \Omega \cup \Gamma_D$  such that  $A \cup B = \Omega \cup \Gamma_D$ , and  $\mathcal{L}^n(B) = |E^s v^1|(A) = 0$ . Then, defining (for  $i = 1, 2$ )

$$\hat{\sigma}_D^i := \begin{cases} \sigma_D^i & \mathcal{L}^n \text{-a.e. in } A, \\ \frac{dEv^1}{d|Ev^1|} & |E^s v^1| \text{-a.e. in } B, \end{cases}$$

it follows that  $\hat{\sigma}_D^1$  and  $\hat{\sigma}_D^2 \in L_{|Ev^1|}^\infty(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n})$ , and

$$\hat{\sigma}_D^1 : \frac{dEv^1}{d|Ev^1|} |Ev^1| = [\sigma_D^1 : Ev^1] = |Ev^1| = [\sigma_D^2 : Ev^1] = \hat{\sigma}_D^2 : \frac{dEv^1}{d|Ev^1|} |Ev^1|.$$

By definition, we have that  $\hat{\sigma}_D^1 = \hat{\sigma}_D^2 |E^s v^1|$ -a.e. in  $\Omega \cup \Gamma_D$ . In addition, taking the absolutely continuous part in (4.1) yields (see [4, 7]),

$$\sigma_D^1 : E^a v^1 = [\sigma_D^1 : Ev^1]^a = |E^a v^1| = [\sigma_D^2 : Ev^1]^a = \sigma_D^2 : E^a v^1.$$

Thus  $\sigma_D^1 = \sigma_D^2 \mathcal{L}^n$ -a.e. in  $\{|E^a v^1| > 0\}$  and finally  $\hat{\sigma}_D^1 = \hat{\sigma}_D^2 |Ev^1|$ -a.e. in  $\Omega \cup \Gamma_D$  as requested.  $\square$

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#### REFERENCES

- [1] G. ANZELLOTTI: On the extremal stress and displacement in Hencky plasticity, *Duke Math. J.* **51**(1) (1984) 133–147.
- [2] J.-F. BABADJIAN: Traces of functions of bounded deformation, *Indiana Univ. Math. J.* **64** (2015) 1271–1290.
- [3] A. BENSOUSSAN, J. FREHSE: Asymptotic behaviour of the time dependent Norton-Hoff law in plasticity theory and  $H^1$  regularity, *Comment. Math. Univ. Carolin.* **37** (1996) 285–304.
- [4] G. DAL MASO, A. DE SIMONE, M. G. MORA: Quasistatic evolution problems for linearly elastic–perfectly plastic materials, *Arch. Rational Mech. Anal.* **180** (2006) 237–291.
- [5] G. DAL MASO, A. DE SIMONE, F. SOLOMBRINO: Quasistatic evolution for Cam-Clay plasticity: a weak formulation via viscoplastic regularization and time rescaling, *Calc. Var. Partial Diff. Eq.* **40** (2011) 125–181.
- [6] F. DEMENGEL, R. TEMAM: Convex function of a measure, *Indiana Univ. Math. J.* **33** (1984) 673–709.
- [7] G.A. FRANCFORT, A. GIACOMINI: Small strain heterogeneous elasto-plasticity revisited, *Comm. Pure Appl. Math.* **65** (2012) 1185–1241.
- [8] V. GIRAULT, P.-A. RAVIART: *Finite element method for Navier-Stokes equations. Theory and algorithm*, Springer-Verlag (1986).
- [9] C. GOFFMAN, J. SERRIN: Sublinear functions of measures and variational integrals, *Duke Math. J.* **31** (1964) 159–178.
- [10] R.V. KOHN, R. TEMAM: Dual spaces of stresses and strains, with applications to Hencky plasticity, *Appl. Math. Optim.* **10** (1983) 1–35.
- [11] A. MAINIK, A. MIELKE: Existence results for energetic models for rate-independent systems, *Calc. Var. PDEs* **22** (2005) 73–99.
- [12] P. SUQUET: Sur les équations de la plasticité: existence et régularité des solutions, *J. Mécanique* **20** (1981) 3–39.
- [13] L. TARTAR: *Topics in nonlinear analysis*, publications mathématiques d’Orsay (1982).
- [14] R. TEMAM: *Mathematical problems in plasticity*, Gauthier-Villars, Paris (1985). Translation of *Problèmes mathématiques en plasticité*, Gauthier-Villars, Paris (1983).

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